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Acoustic scattering by density gradients

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In this paper we solve some model problems by the method of matched expansions and deduce some features of Green's function using the reciprocal theorem to demonstrate that scattering by density gradients can enhance the radiation efficiency of quadrupoles to that of dipoles. If a fluid is subjected to body forces close to and on both sides of a density interface, then the source system degenerates to a quadrupole only if the components of the *forces* perpendicular to the interface are equal in magnitude and opposite in direction and, additionally, the components of the *forces per unit fluid density* parallel to the interface are equal but opposite. In other cases dipole rather than quadrupole radiation efficiency is achieved. But we also show that, despite published assertions to the contrary, the radiation efficiency is not enhanced further and there is no monopole contribution to the radiated sound.

1. Introduction

Lighthill's (1952, 1954) theory of aerodynamic sound generation underpredicts the noise of aeroengines at low jet Mach numbers. The radiated sound is dominated by the contribution from Lighthill's quadrupoles at high Mach numbers, but as the jet velocity is reduced other sources of sound evidently become important. These sources, which radiate sound that increases with the jet velocity slowly compared with quadrupole noise, might result from scattering mechanisms that enhance the radiation efficiency of Lighthill's quadrupoles, or they might be entirely new dipole or monopole sources.

Solid bodies can act as such efficient scatterers of sound since the unsteady forces induced on the fluid constitute acoustic dipoles. Inhomogeneities in density behave similarly; the coupling of entropy and acoustic waves in flows through nozzles and turbines has been modelled by Candel (1972) and Cumpsty & Marble (1974), and Ffowcs Williams & Howe (1975) have shown that, when a 'pellet' or 'slug' of fluid of different density is accelerated with the mean jet flow through a nozzle contraction, the unsteady force radiates sound with dipole efficiency. Ffowcs Williams (1975) maintains further that in these situations the scattering can be even more effective and that monopole sources as well as dipoles can be present; he argues that a 'monopole splash' occurs when rigid bodies traverse density gradients, and it has been suggested that a similar monopole will occur owing to variations in the speed of sound. We shall show, however, that no such monopoles exist.

Inhomogeneities in density are also thought to be important in scattering the noise of hot low-speed jets. Hoch *et al.* (1973), Tanna, Fisher & Dean (1973) and Tanna, Dean & Fisher (1975) have determined the effect that the temperature of a model jet

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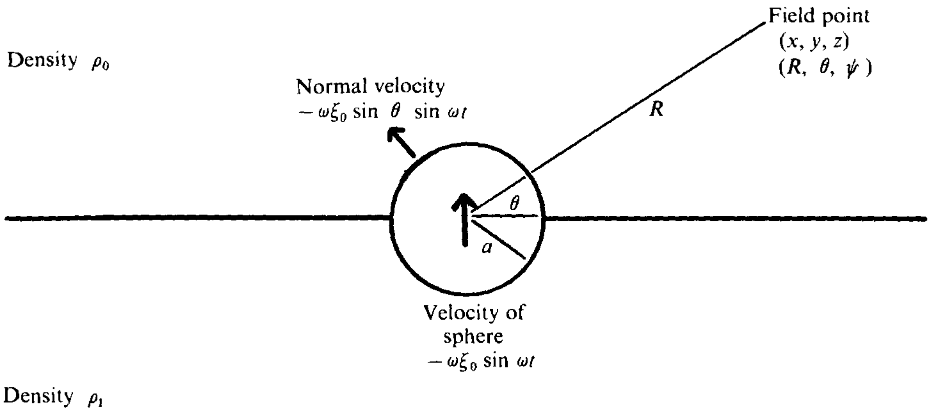

FIGURE 1. Sphere vibrating at the interface of two fluids.

has on the radiated sound. They found that at high speeds a hot jet was quieter than a cold jet, but at low speeds it was noisier. As the temperature of the jet is increased, the density is reduced and with it the quadrupole strength, so it is not surprising that the noise at high speeds is reduced. But at low speeds the noise is increased; this increase is believed to be because scattering by density gradients enhances the radiation efficiency of the quadrupoles.

Green's function techniques (Tester & Morfey 1976) and reformulations of Lighthill's acoustic analogy (Morfey 1973; Ffowcs Williams 1974; Howe 1975) show that dipole radiation results from such scattering of quadrupole near fields by density gradients, and Tester & Morfey (1976) and Morfey & Szewczyk (1977) have obtained empirical correlations of hot-jet noise data assuming just quadrupole and dipole radiation. Mani (1976), on the other hand, claimed that the radiation efficiency could be increased still further to that of monopoles, and both he and Lush & Fisher (1973) have obtained a good collapse of the data assuming that monopoles are also present. We shall show, however, that there are no such monopoles and that Mani was mistaken to include them in his plug-flow modelling of the jet. (Of course the absence of monopoles does not affect the good agreement with experiment obtained by Mani at high speeds.) Lush & Fisher (1973) and others have already shown there is no such monopole if the inhomogeneity in density is compact on a wavelength scale. We shall show this is still true if the region where the density varies is extensive.

We show in this paper that scattering by density gradients does not induce a monopole source. We examine some model problems to illustrate that no 'monopole splash' occurs though density gradients can scatter the near field of quadrupoles and enhance their radiation efficiency to that of dipoles. We show that the field of a *monopole* located close to a discontinuity in the fluid density or speed of sound is the same regardless of whether the monopole is just above or just below the interface, and that similar conclusions hold for a *dipole* with axis parallel to the discontinuity but not for one with axis perpendicular to it. We also examine Mani's arguments for the existence of monopoles in the mixing of turbulent jets and show them to be in error.

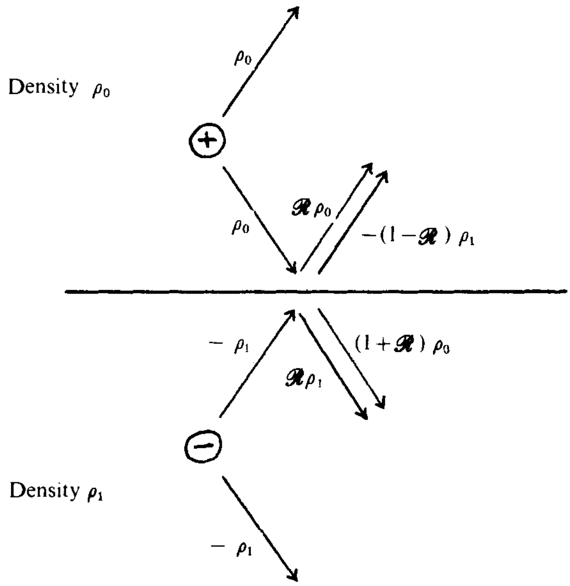

FIGURE 2. The direct, reflected and transmitted fields from a dipole at a density interface. $\rho_0+\mathscr{R}\rho_0-(1-\mathscr{R})\,\rho_1 = -\rho_1+\mathscr{R}\rho_1+(1+\mathscr{R})\,\rho_0 = 0$.

2. Dipole sources near a density interface

We shall demonstrate that a compact sphere vibrating without deformation at a density interface constitutes an acoustic dipole. We consider a sphere of radius a which vibrates such that its centre is located on the y axis at a distance $\xi_0 \cos \omega t$ above the origin; the amplitude of the oscillation ξ_0 is small enough for the governing equation and boundary conditions to be linearized. We assume that the density of the fluid is ρ_0 for $y > 0$ and ρ_1 for $y < 0$, but in order to model the effect of density gradients alone we assume, like Ffowcs Williams (1975), that the speed of sound c is uniform. The geometry is illustrated in figure 1.

It is surprisingly easy to determine the radiated sound if we solve for the velocity potential ϕ rather than the pressure p. The velocity potential satisfies the wave equation

$$\partial^2\phi/\partial t^2 - c^2\nabla^2\phi = 0, \tag{1}$$

the radiation condition, and the boundary conditions (i) normal velocity specified on the sphere,

$$\partial\phi/\partial R = -\omega\xi_0 \sin\omega t \sin\theta \quad \text{on} \quad R = a, \tag{2}$$

(ii) pressure continuous across the density interface,

$$\rho_0\,\partial\phi(y=0^+)/\partial t = \rho_1\,\partial\phi(y=0^-)/\partial t, \tag{3}$$

and (iii) normal velocity continuous across the interface,

$$\partial\phi(y=0^+)/\partial y = \partial\phi(y=0^-)/\partial y. \tag{4}$$

The solution is simply

$$\phi = \mathrm{Re}\left\{\frac{\partial}{\partial y}\left(\frac{i\omega\xi_0 a^3 \exp\{i\omega t - i\omega(R-a)/c\}}{2R(1+i\omega a/c - \frac{1}{2}\omega^2a^2/c^2)}\right)\right\}, \tag{5}$$

where Re denotes the real part. (The boundary condition of continuity of pressure is automatically satisfied since ϕ vanishes on $y = 0$.) We find that the far-field fluctuating pressure is

$$p = \begin{cases} \mathrm{Re}\left\{\dfrac{i\omega^3\xi_0 a^3\rho_0 \sin\theta \exp\{i\omega t - i\omega(R-a)/c\}}{2cR(1+i\omega a/c - \frac{1}{2}\omega^2a^2/c^2)}\right\}, & y > 0, \quad (6a) \\ \mathrm{Re}\left\{\dfrac{i\omega^3\xi_0 a^3\rho_1 \sin\theta \exp\{i\omega t - i\omega(R-a)/c\}}{2cR(1+i\omega a/c - \frac{1}{2}\omega^2a^2/c^2)}\right\}, & y < 0. \quad (6b) \end{cases}$$

The sphere acts as a dipole source of sound; there is no 'monopole splash'.

The reflexion properties of a density interface suggest that this result is also true for asymmetric vibrating bodies (see figure 2). There are contributions to the radiated sound from one monopole above and from another of opposite strength below the interface. The pressure field radiated in the upper half-plane directly by the first monopole is proportional to ρ_0, while the fields reflected by and transmitted through the interface are proportional to $\mathscr{R}\rho_0$ and $(1+\mathscr{R})\rho_0$ respectively. The reflexion coefficient $\mathscr{R}$ is given by $\mathscr{R} = (\rho_1-\rho_0)/(\rho_1+\rho_0)$. The second monopole radiates directly in the lower half-plane a pressure field proportional to $-\rho_1$, and because the reflexion coefficient now has opposite sign, the reflected and transmitted fields are proportional to $\mathscr{R}\rho_1$ and $-(1-\mathscr{R})\rho_1$ respectively. Since $\rho_0+\mathscr{R}\rho_0-(1-\mathscr{R})\rho_1 = 0$, the monopole contribution to the total field vanishes (see figure 2).

Ffowcs Williams' (1975) result is in error because he neglects what is actually an important contribution from the isotropic quadrupole $\nabla^2(p-c^2\rho)$, equivalent to a monopole $c^{-2}\,\partial^2(p-c^2\rho)/\partial t^2$. The contribution from the term $c^{-2}\,\partial^2 p/\partial t^2$ is smaller by a factor of order M^2, the mean-flow Mach number squared, than the contribution from the term $-\partial^2\rho/\partial t^2$, and so we neglect it. Thus in Ffowcs Williams' notation (ρ^α, ρ^β, $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$ are the densities and velocities of the α and β fluids, D_α/Dt is the time derivative following 'α' particles, and β is the concentration of 'β' fluid), the monopole source has strength not simply

$$\int Q\,dV,$$

but

$$\int\left(Q-\frac{\partial\rho^\alpha}{\partial t}\right)dV,$$

where $Q = -\rho^\alpha D_\alpha \ln(1-\beta)/Dt$ and the integration is to be performed over the source region. We can then assume that the 'α' fluid is incompressible in the source region,

$$D_\alpha\rho^\alpha/Dt = 0,$$

and use 'β' mass conservation,

$$\frac{\partial}{\partial t}(\beta\rho^\beta)+\frac{\partial}{\partial x_i}(\beta\rho^\beta u_i^\beta) = 0$$

and

$$\rho^\beta = \text{constant},$$

to determine the monopole source strength as

$$\int\left(\frac{-\rho^\alpha}{(1-\beta)}\frac{\partial}{\partial x_i}(\beta u_i^\beta)+\frac{\rho^\alpha u_i^\alpha}{(1-\beta)}\frac{\partial\beta}{\partial x_i}+u_i^\alpha\frac{\partial\rho^\alpha}{\partial x_i}\right)dV = \int\{(1-\beta)\,u_i^\alpha+\beta u_i^\beta\}\frac{\partial}{\partial x_i}\left(\frac{\rho^\alpha}{1-\beta}\right)dV. \quad (7)$$

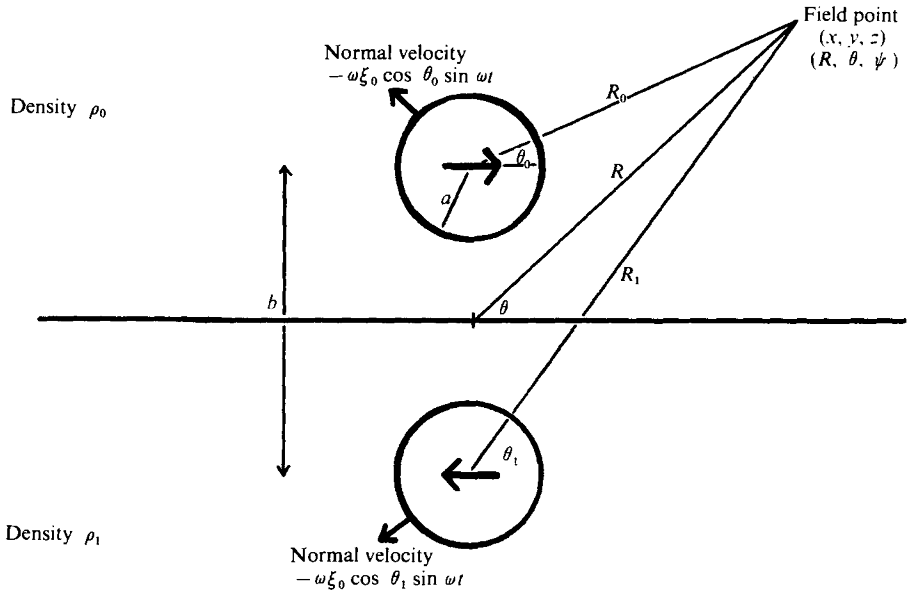

FIGURE 3. Spheres vibrating with velocities parallel to a density interface.

There will be a compensatory back-flow of the 'α' fluid as the 'β' particles move through, with
$$\{(1-\beta)\,u_i^{\alpha}+\beta u_i^{\beta}\} = 0$$
for incompressible flow. This results in the monopole source strength (7) vanishing. So the inclusion of the contribution from the isotropic quadrupole ensures that sound is radiated with the efficiency only of dipoles; not all quadrupoles are negligible in comparison with monopoles.

3. Quadrupole sources near a density interface

We now show that two compact vibrating spheres situated close to and on opposite sides of a density interface constitute a quadrupole source if their velocities, equal in magnitude and opposite in direction, are parallel to the interface. The forces on the spheres are different in magnitude, but the forces per unit fluid density are equal. The spheres both have radius a, and their centres are located at distances $\frac{1}{2}b$ above and below the x axis and $\xi_0 \cos \omega t$ to the right and left of the y axis; the distance between the spheres is much smaller than a wavelength, $\omega b/c \ll 1$, and the amplitude of the oscillation ξ_0 is small enough for the governing equation and boundary conditions to be linearized. (R_0, θ_0, ψ), (R_1, θ_1, ψ) and (R, θ, ψ) are spherical polar co-ordinates with origins respectively at the centres of the upper and lower spheres and at the midpoint between the spheres (see figure 3).

Since the source region is compact on a wavelength scale, we determine the source structure by assuming that the flow is incompressible. The velocity potential ϕ satisfies
$$\nabla^2\phi = 0 \qquad (8)$$

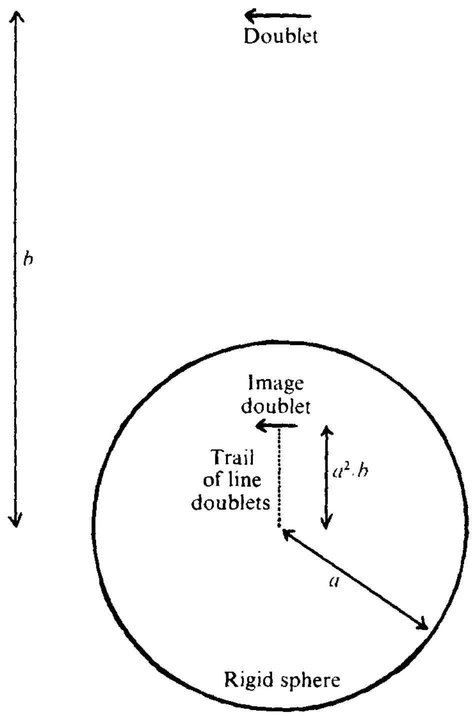

FIGURE 4. Image system for a doublet near a rigid stationary sphere.

with boundary conditions (i) normal velocity specified on the spheres,

$$\left.\begin{aligned}\partial\phi/\partial R_0 &= -\omega\xi_0 \sin\omega t \cos\theta_0 \sin\psi \quad \text{on} \quad R_0 = a,\\ \partial\phi/\partial R_1 &= \omega\xi_0 \sin\omega t \cos\theta_1 \sin\psi \quad \text{on} \quad R_1 = a,\end{aligned}\right\} \tag{9}$$

(ii) pressure continuous across the interface,

$$\rho_0 \,\partial\phi(y = 0^+)/\partial t = \rho_1 \,\partial\phi(y = 0^-)/\partial t, \tag{10}$$

and (iii) normal velocity continuous across the interface,

$$\partial\phi(y = 0^+)/\partial y = \partial\phi(y = 0^-)/\partial y. \tag{11}$$

We find that the solution for the velocity potential satisfies $\partial\phi/\partial t = 0$ on $y = 0$, so that the requirement of continuity of pressure (ii) is automatically satisfied; i.e. there is no field scattered by the density interface.

We can represent the flow field as the field of an infinite number of doublets which lie on the line joining the centres of the spheres and which have their axes in the x direction. If we neglect the scattering by one sphere of the field from the other, just two doublets of opposite sign suffice. But when scattering is incorporated an infinite series of image doublets and line doublets is needed. To show this we use Hicks' (1880) result for the change in the field of a doublet when a rigid stationary sphere is introduced (see also Lamb 1932, §§98 and 99). If the line joining the doublet to the sphere's centre is perpendicular to the doublet's axis, the image system is a second doublet at the inverse point with a trail of line doublets behind it (see figure 4). So, to a first approximation, a

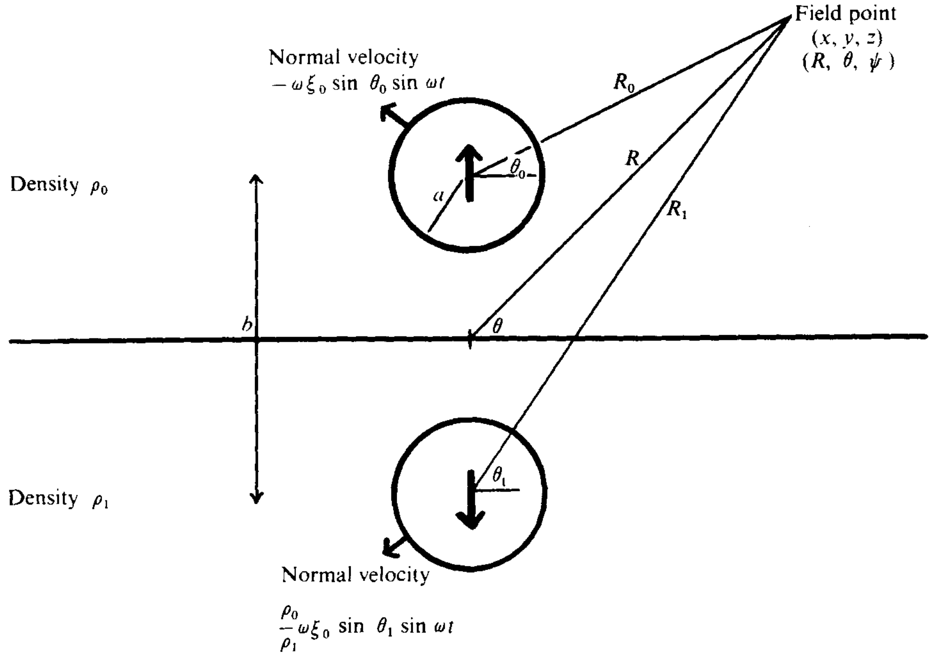

FIGURE 5. Spheres vibrating with velocities perpendicular to a density interface.

second stationary sphere augments the field of a vibrating sphere by introducing an image doublet and a trail of line doublets. But this scattered field is itself scattered by the first sphere, and so the interaction of the two spheres results in an infinite series of image doublets. When both spheres are vibrating, the doublets occur in pairs at equal distances above and below the origin, and when the spheres have equal but opposite velocities, these doublets have equal but opposite strengths. Since the characteristics of the flow field are the same for each pair of doublets, we need consider only the field radiated directly by the spheres. The scattered field has the same multipole nature but is smaller at large distances by a factor of order a^3/b^3. So for large R_0 and R_1,

$$\begin{aligned}\phi &= \tfrac{1}{2}\omega\xi_0 a^3 \sin\omega t\left(\frac{\cos\theta_0\sin\psi}{R_0^2}-\frac{\cos\theta_1\sin\psi}{R_1^2}\right)\left(1+O\left(\frac{a^3}{b^3}\right)\right)\\ &= \tfrac{1}{2}\omega\xi_0 a^3 b\sin\omega t\left(\frac{\partial^2}{\partial x\,\partial y}\frac{1}{R}+\frac{b^2}{24}\frac{\partial^4}{\partial x\,\partial y^3}\frac{1}{R}+\dots\right)\left(1+O\left(\frac{a^3}{b^3}\right)\right).\end{aligned} \quad (12)$$

We can in principle determine the radiated sound by matching on an outgoing solution to the wave equation which satisfies the boundary conditions of continuity of pressure and displacement across the interface. The detailed calculation of the field would be laborious, but it is evident from (12) that sound is radiated only with the efficiency of quadrupoles.

There is a residual dipole, however, if the velocities of the spheres are perpendicular to the interface. Then the source degenerates to a quadrupole when the magnitudes of the *forces* on the spheres are equal, rather than their *speeds*. To demonstrate this we assume that the centres of the spheres are located at distances $\frac{1}{2}b+\xi_0\cos\omega t$ above and

$\frac{1}{2}b+(\rho_0/\rho_1)\,\xi_0\cos\omega t$ below the origin. The boundary conditions of normal velocity specified on the spheres become

$$\left.\begin{aligned}\partial\phi/\partial R_0 &= -\omega\xi_0\sin\omega t\sin\theta_0 \quad \text{on} \quad R_0=a,\\ \partial\phi/\partial R_1 &= \frac{\rho_0}{\rho_1}\omega\xi_0\sin\omega t\sin\theta_1 \quad \text{on} \quad R_1=a.\end{aligned}\right\} \tag{13}$$

The geometry is illustrated in figure 5.

To ensure that the pressure is continuous across the density interface, we now need an additional term in the velocity potential. If we neglect the scattering by one sphere of the field from the other, we find that

$$\phi = \tfrac{1}{2}\omega\xi_0 a^3\sin\omega t\left(\frac{\sin\theta_0}{R_0^2}-\frac{\rho_0}{\rho_1}\frac{\sin\theta_1}{R_1^2}-\frac{(\rho_1-\rho_0)\,(\frac{1}{2}b+|y|)\,\mathrm{sgn}\,y}{\rho_1\{x^2+(\frac{1}{2}b+|y|)^2+z^2\}^{\frac{3}{2}}}\right). \tag{14}$$

The field in $y>0$ is the field of a doublet with its axis in the y direction situated a distance $\frac{1}{2}b$ above the origin and one of opposite sign situated a distance $\frac{1}{2}b$ below the origin. For the field in $y<0$ the strengths of the doublets are multiplied by the factor ρ_0/ρ_1. Scattering by the spheres introduces an infinite series of image doublets which lie on the line joining the centres of the spheres and which have their axes in the y direction. The doublets occur in pairs at equal distances above and below the origin, with their strengths in the ratio 1 to ρ_0/ρ_1. Again the characteristics of the flow field are the same for each pair of doublets, so for large R

$$\phi = \begin{cases} -\frac{1}{2}\omega\xi_0 a^3 b\sin\omega t\left(\dfrac{\partial^2}{\partial y^2}\dfrac{1}{R}+\dfrac{b^2}{24}\dfrac{\partial^4}{\partial y^4}\dfrac{1}{R}+\ldots\right)\left(1+O\left(\dfrac{a^3}{b^3}\right)\right), & y>0, \quad (15a)\\ -\frac{1}{2}\omega\xi_0 a^3 b\dfrac{\rho_0}{\rho_1}\sin\omega t\left(\dfrac{\partial^2}{\partial y^2}\dfrac{1}{R}+\dfrac{b^2}{24}\dfrac{\partial^4}{\partial y^4}\dfrac{1}{R}+\ldots\right)\left(1+O\left(\dfrac{a^3}{b^3}\right)\right), & y<0. \quad (15b)\end{cases}$$

Sound is again radiated with the efficiency only of quadrupoles.

4. Sources near gradients in the density and speed of sound

The results of §§2 and 3 also hold if the speed of sound differs in the two fluids. We use the reciprocal theorem to demonstrate this; it can also be deduced directly from the governing equations, or from the work of Tester & Morfey (1976).

Landau & Lifshitz (1959, p. 288) established a reciprocal theorem for sound waves in inhomogeneous media at rest when the fluid density and speed of sound vary. They showed that the pressure at a field point B due to a point volume monopole source at A is equal to the pressure at A due to an identical monopole source at B. For an observer at A close to a discontinuity in the density and the speed of sound the pressure due to a monopole source at B is independent of the precise location of A since pressure is continuous across a fluid interface. Application of the reciprocal theorem therefore shows that the pressure at B due to a monopole source at A is also independent of the precise location of A; the field is the same regardless of which side of the interface the source is located.

This reciprocal principle also demonstrates that a compact rigid body vibrating in any manner does not constitute an acoustic monopole regardless of any variations in the mean density or speed of sound, and this result can be generalized immediately to

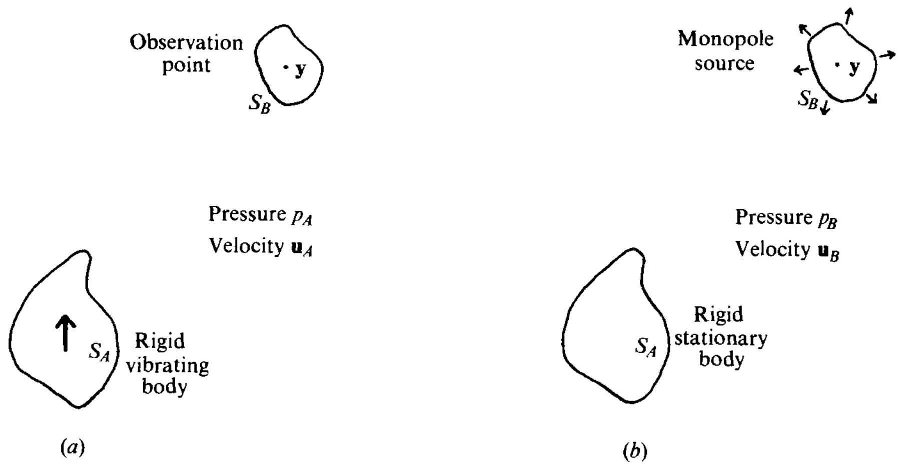

FIGURE 6. (*a*) The direct and (*b*) the reciprocal problem

$$p_A(\mathbf{y}) = \int_{S_A} p_B(\mathbf{x}) \frac{\partial \mathbf{u}_A}{\partial t}(\mathbf{x}) . \mathbf{dS}(\mathbf{x}) \Big/ \frac{\partial}{\partial t} \int_{S_B} \mathbf{u}_B(\mathbf{x}') . \mathbf{dS}(\mathbf{x}').$$

more than one compact rigid body. Here we restrict attention to fluctuations of frequency ω in an ideal fluid, and denote by p_A and $\mathbf{u}_A$ the fluctuating pressure and velocity field due to small vibrations of a rigid body in an inhomogeneous medium at rest. We denote by p_B and $\mathbf{u}_B$ the fluctuating pressure and velocity in the reciprocal problem when a point source is at $\mathbf{y}$ in the same medium and the rigid body is stationary (see figure 6). We shall show that there is no monopole contribution to p_A by deriving an expression for $p_A(\mathbf{y})$ which involves the integral of p_B over the rigid body and which vanishes if differences in retarded times across the rigid body are neglected.

Both p_A and p_B satisfy (see Landau & Lifshitz 1959, equation 74.2)

$$\nabla . \left(\frac{\nabla p}{\rho}\right) + \frac{\omega^2 p}{\rho c^2} = 0, \tag{16}$$

so
$$\nabla . \left(\frac{p_A \nabla p_B}{\rho} - \frac{p_B \nabla p_A}{\rho}\right) = p_A \nabla . \left(\frac{\nabla p_B}{\rho}\right) - p_B \nabla . \left(\frac{\nabla p_A}{\rho}\right) = 0.$$

We integrate this equation over a volume that is bounded by S_A, the surface of the rigid body, by S_B, a small sphere that encloses the point $\mathbf{y}$, and by S_∞, an infinitely distant closed surface, and then transform the volume integral into three surface integrals to obtain

$$\int_{S_\infty} \left(\frac{p_A \nabla p_B}{\rho} - \frac{p_B \nabla p_A}{\rho}\right) . \mathbf{dS} + \int_{S_A} \left(\frac{p_A \nabla p_B}{\rho} - \frac{p_B \nabla p_A}{\rho}\right) . \mathbf{dS} + \int_{S_B} \left(\frac{p_A \nabla p_B}{\rho} - \frac{p_B \nabla p_A}{\rho}\right) . \mathbf{dS} = 0. \tag{17}$$

At large distances R, the integrand $(p_A \nabla p_B - p_B \nabla p_A)/\rho$ decays faster than R^{-2}, so the integral over S_∞ vanishes. Near $\mathbf{y}$ the pressure p_B varies rapidly so there $\nabla p_B/p_B$ is much greater than $\nabla p_A/p_A$, and the integral over S_B may be approximated by

$$\int_{S_B} \frac{p_A \nabla p_B}{\rho} . \mathbf{dS}.$$

Furthermore p_A is nearly constant over S_B. So in the limit when the radius of S_B tends to zero, the surface integral over S_B is given by

$$p_A(\mathbf{y})\int_{S_B}\frac{\nabla p_B}{\rho}\,.\,\mathbf{dS} = -p_A(\mathbf{y})\frac{\partial}{\partial t}\int_{S_B}\mathbf{u}_B\,.\,\mathbf{dS}.$$

Finally, the integral over S_A is equal to

$$\int_{S_A}\left(-p_A\frac{\partial \mathbf{u}_B}{\partial t}+p_B\frac{\partial \mathbf{u}_A}{\partial t}\right).\,\mathbf{dS} = \int_{S_A}p_B\frac{\partial \mathbf{u}_A}{\partial t}.\,\mathbf{dS},$$

since $\mathbf{u}_B\,.\,\mathbf{dS}$ vanishes on S_A. Substituting these results into (17) we obtain

$$p_A(\mathbf{y}) = \int_{S_A}p_B(\mathbf{x})\frac{\partial \mathbf{u}_A}{\partial t}(\mathbf{x})\,.\,\mathbf{dS}(\mathbf{x})\Big/\frac{\partial}{\partial t}\int_{S_B}\mathbf{u}_B(\mathbf{x}')\,.\,\mathbf{dS}(\mathbf{x}'). \tag{18}$$

Since

$$\int_{S_A}\frac{\partial \mathbf{u}_A}{\partial t}.\,\mathbf{dS} = 0,$$

variations in $p_B(\mathbf{x})$ across S_A are solely responsible for the non-vanishing of $p_A(\mathbf{y})$. But, since pressure is continuous across interfaces in the density and speed of sound, these variations occur only because of differences in the retarded time across S_A. So the vibrating body radiates sound only because of differences in the retarded time across it; it does not constitute a monopole source.

The field of a dipole close to a density interface that is locally perpendicular to the dipole axis *does* depend on the precise location of the dipole, since the normal gradient of pressure is not continuous across a density interface. Consequently density gradients can scatter the near fields of quadrupoles and enhance the radiation efficiency to that of dipoles.

5. Applications to the noise of hot jets

We have demonstrated that scattering by a density interface does not induce a monopole source. Yet Mani (1976) maintains that monopoles are present when a hot jet mixes with its cold surroundings, even in the absence of heat diffusion and viscous dissipation. These monopoles arise because of the classification he uses for the source terms of Lilley's equation:

$$\begin{aligned}\mathscr{L}(p') &\equiv \frac{1}{\bar{c}^2}\frac{\bar{D}^3p'}{\bar{D}t^3}-\frac{\bar{D}}{\bar{D}t}\nabla^2p'-\frac{d}{dr}\log\bar{c}^2\,\frac{\bar{D}}{\bar{D}t}\frac{\partial p'}{\partial r}+2\frac{dV_1}{dr}\frac{\partial^2p'}{\partial x\,\partial r}\\ &\approx \bar{\rho}\frac{\bar{D}}{\bar{D}t}\left(\frac{\partial^2(u_i'u_j')}{\partial x_i\,\partial x_j}\right),\end{aligned} \tag{19}$$

where $\bar{D}/\bar{D}t = \partial/\partial t+V_1\,\partial/\partial x$, p' and $\mathbf{u}'$ are the perturbation pressure and velocity, $\bar{\rho}$ and $\bar{c}^2$ are the mean density and the mean square of the speed of sound, and $V_1(r)$ is the mean velocity in the x direction. The mean flow is assumed to be axisymmetric (r is the radial distance in cylindrical polar co-ordinates) and independent of x, the distance along the jet axis.

Mani asserts that a source term of the form

$$\bar{\rho}(r)\,\partial^2[\delta(\mathbf{x}-\mathbf{x}_0)]/\partial x_i\,\partial x_j$$

contains a quadrupole term

$$\bar{\rho}(r_0)\,\partial^2[\delta(\mathbf{x}-\mathbf{x}_0)]/\partial x_i\,\partial x_j,$$

dipole terms

$$-\frac{\partial\bar{\rho}}{\partial x_i}(r_0)\frac{\partial}{\partial x_j}\delta(\mathbf{x}-\mathbf{x}_0)-\frac{\partial\bar{\rho}}{\partial x_j}(r_0)\frac{\partial}{\partial x_i}\delta(\mathbf{x}-\mathbf{x}_0)$$

and a monopole term

$$\frac{\partial^2\bar{\rho}}{\partial x_i\,\partial x_j}(r_0)\,\delta(\mathbf{x}-\mathbf{x}_0).$$

But we do not find this classification helpful since it does not reflect the efficiency with which different sources radiate. We believe that the significant multipole characteristics of the source are embodied in the factor that multiplies the mean density, $\bar{\rho}(r)$ not $\bar{\rho}(r_0)$, in the form of Lilley's equation used by Mani. For a point volume–velocity source of strength $\delta(\mathbf{x}-\mathbf{x}_0)\cos\omega t$ in the continuity equation

$$\rho^{-1}D\rho/Dt+\nabla.\mathbf{u}=\delta(\mathbf{x}-\mathbf{x}_0)\cos\omega t$$

appears as

$$\bar{\rho}(r)\frac{\bar{D}}{\bar{D}t}\frac{D}{Dt}[\delta(\mathbf{x}-\mathbf{x}_0)\cos\omega t]$$

in (19); two adjacent sources with strengths $\delta(\mathbf{x}-\mathbf{x}_0)\cos\omega t$ and $-\delta(\mathbf{x}-\mathbf{x}_0-\mathbf{l})\cos\omega t$ appear as

$$\bar{\rho}(r)\frac{\bar{D}}{\bar{D}t}\frac{D}{Dt}\left(l_i\frac{\partial}{\partial x_i}\delta(\mathbf{x}-\mathbf{x}_0)\cos\omega t\right),$$

not as

$$\bar{\rho}(r_0)\frac{\bar{D}}{\bar{D}t}\frac{D}{Dt}\left(l_i\frac{\partial}{\partial x_i}\delta(\mathbf{x}-\mathbf{x}_0)\cos\omega t\right)$$

nor as

$$l_i\frac{\partial}{\partial x_i}\left(\bar{\rho}(r)\frac{\bar{D}}{\bar{D}t}\frac{'D}{Dt}\,\delta(\mathbf{x}-\mathbf{x}_0)\cos\omega t\right).$$

Consequently we agree with Tester & Morfey (1976), who interpret the term

$$\bar{\rho}\frac{\bar{D}}{\bar{D}t}\left(\frac{\partial^2(u_i'u_j')}{\partial x_i\,\partial x_j}\right)$$

as a volume–acceleration distribution

$$\partial^2(u_i'u_j')/\partial x_i\,\partial x_j$$

of quadrupole order; the radiation efficiency of these quadrupoles can be enhanced by density gradients to that of dipoles.

Mani next expresses the radiated pressure as

$$p'(x)=\iiiint u_i'u_j'\left(\bar{\rho}\frac{\partial^2 g}{\partial y_i\,\partial y_j}+\frac{\partial\bar{\rho}}{\partial y_i}\frac{\partial g}{\partial y_j}+\frac{\partial\bar{\rho}}{\partial y_j}\frac{\partial g}{\partial y_i}+\frac{\partial^2\bar{\rho}}{\partial y_i\,\partial y_j}g\right)dV_y\,d\tau, \tag{20}$$

where Green's function g satisfies

$$\mathscr{L}(g)\;=\;\bar{D}[\delta(\mathbf{x}-\mathbf{y})\,\delta(t-\tau)]/\bar{D}t,$$

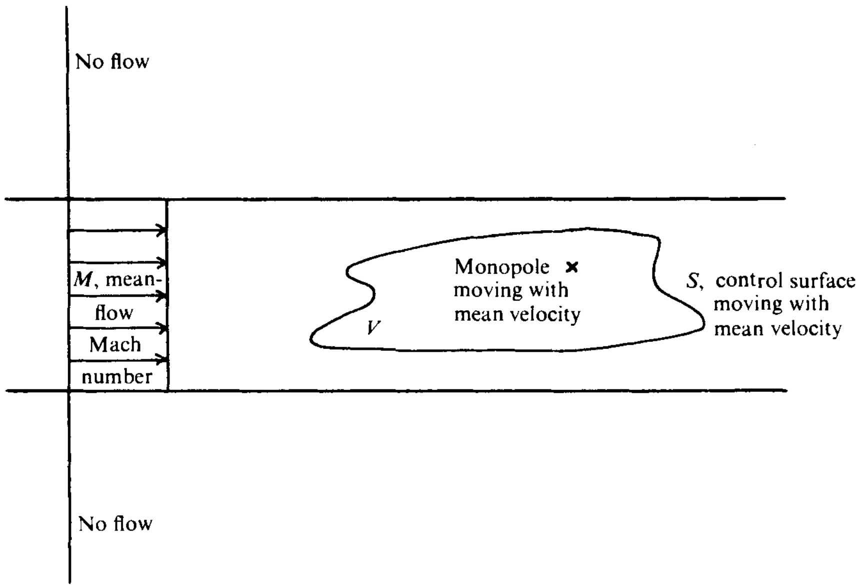

FIGURE 7. Mani's plug-flow modelling of the jet.

and interprets the terms in $\partial g/\partial y_i$, $\partial g/\partial y_j$ and g as being of lower order than the term in $\partial^2 g/\partial y_i\, \partial y_j$. But this interpretation does not provide an ordering from which far-field estimates of the radiated sound can immediately be deduced. Such an ordering allows meaningful comparisons of the radiated sound only if a length scale for the gradient operator can be estimated unambiguously. Derivatives with respect to the source's position $\mathbf{y}$ can be expressed as derivatives with respect to the observer's position $\mathbf{x}$ when g is a function of $\mathbf{x}-\mathbf{y}$ alone. So then the length scale for the gradient operator is given unambiguously by the acoustic wavelength. But in the case considered by Mani there is no simple estimate for this length scale. The contributions to the radiated sound from the terms involving $\partial g/\partial y_i$, $\partial g/\partial y_j$ and $\partial^2 g/\partial y_i\, \partial y_j$ are not smaller by factors of the order of the mean-flow Mach number than the contribution from the term in g. Indeed an alternative manipulation of Lilley's equation (Tester & Morfey 1976) shows that they combine to cancel it to lowest order in the Mach number.

Finally Mani draws support for the existence of monopoles from his belief that a point source radiates a pressure proportional to the local density at the source and that this 'leads to the generation of lower-order singularities when density gradients are considered'. But the reciprocal theorem used in §4 shows this belief to be unfounded.

We consequently have doubts about the usefulness of Mani's classification of the sources of noise in hot jets; and we believe he is wrong when he employs a different modelling of the jet (a plug-flow modelling) with a different definition for the multipole nature of the source, but still assumes the existence of monopoles. Consider a compact control surface S inside the jet that moves with the mean velocity and encloses a volume V containing the monopole source (see figure 7). The moving monopole, with strength of the order of the mean-flow Mach number M, requires the mass flux through S to scale with M. But, in a frame of reference moving with the mean flow, the continuity equation,

$$D\rho/Dt = -\rho\nabla.\mathbf{u}, \tag{21}$$

the equation of state,

$$p = \mathrm{R}\rho T, \tag{22}$$

the equation for isentropic flow,

$$\rho\, Ds/Dt = 0, \tag{23}$$

and the equation relating the thermodynamic quantities,

$$T ds = c_v dT + p d(1/\rho), \tag{24}$$

together with an application of the divergence theorem imply that the mass flux through such a surface will scale as M^3. For

$$\int_S \mathbf{u} . \mathbf{dS} = \int_V \nabla . \mathbf{u} dV = -\int_V \frac{1}{\rho}\frac{D\rho}{Dt} dV = -\int_V \frac{1}{\gamma p}\frac{Dp}{Dt} dV,$$

and pressure variations in the source region scale as ρU^2. Here R is the specific gas constant, T the temperature, s the specific entropy, c_v the specific heat at constant volume, γ the ratio of specific heats, and U the mean-flow velocity. Mani's description of the sources in his plug-flow modelling of the jet is therefore inappropriate. No monopole source is present.

6. Conclusions

Experiments on model jets (Hoch *et al.* 1973; Tanna, Fisher & Dean 1973; Tanna, Dean & Fisher 975) show that at low Mach numbers an increase in the jet temperature results in an increase in the radiated sound. Empirical correlations of this 'excess noise' have been obtained (Lush & Fisher 1973) under the assumption of the existence of monopole sources in the jet, and it has been suggested that the scattering of quadrupole near fields by density gradients could generate such monopoles, even in an ideal fluid. But this is incorrect. No monopole sources are present, though the scattering can enhance the radiation efficiency of the quadrupoles to that of dipoles.

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